

ON A PROBLEM OF BÉRARD-BERGERY AND BESSE

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ABSTRACT. We show that every Eells-Kuiper quaternionic projective plane carries a Riemannian metric such that all geodesics passing through a certain point are simply closed and of the same length. This result resolves a longstanding problem of Bérard-Bergery and Besse dating back to 1970's.

1. INTRODUCTION

Let p be a point in a closed manifold M . Let g be a Riemannian metric on M . The Riemannian structure (M, g) is called an SC^p Riemannian structure if all geodesics issued from p are simply closed (periodic) geodesics with the same length. We refer to the classic book [Be] for a systematic account of the SC^p structures.

It is clear that there are SC^p Riemannian structures on the compact symmetric spaces of rank one (briefed in [Be] as CROSS), namely the unit spheres, the real projective spaces, the complex projective spaces, the quaternionic projective spaces and the Cayley projective plane, endowed with the corresponding canonical metrics. Moreover, a fundamental result of Bott [Bo] states that any smooth manifold carrying an SC^p structure should have the same integral cohomology ring as that of a CROSS. On the other hand, there are manifolds verifying the above cohomological condition but not diffeomorphic to any CROSS. For typical examples, we mention the (exotic) homotopy spheres and the Eells-Kuiper (exotic) quaternionic projective planes.

In 1975, Bérard-Bergery [BB] discovered an SC^p structure on an exotic sphere of dimension 10. He then raised the natural question: *is there any (exotic) Eells-Kuiper quaternionic projective plane carrying an SC^p structure?* The same question was also posed explicitly by Besse in the classic book [Be, 0.15 on pp. 4]. Moreover, it is pointed out in [Be, pp. 143] that a positive answer to the above question would also give a positive nontrivial example to the following open question: *whether a Blaschke manifold at a point¹ would carry an SC^p Riemannian structure?*

The purpose of this article is to provide a positive answer to the above two questions concerning the Eells-Kuiper quaternionic projective planes.

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¹Cf. [Be, 5.37 on page 135] for a definition.

Before going on, we describe the Eells-Kuiper quaternionic projective planes as follows, starting with the standard construction of Milnor [Mi1].

For any pair of integers (h, j) , let $\xi_{h,j}$ be the S^3 -bundle over S^4 determined by the characteristic map $f_{h,j} : S^3 \rightarrow SO(4)$ with $f_{h,j}(u)v = u^h v u^j$ for $u \in S^3$, $v \in \mathbf{R}^4$, where we identify \mathbf{R}^4 with the space of quaternions. It is shown in [Mi1] that when $h + j = 1$, the total space of the above sphere bundle is homeomorphic to the unit sphere S^7 . From now on, we denote by M_h this total space corresponding to $(h, j) = (h, 1 - h)$, and denote by N_h the associated disk bundle.

Remark 1.1. When $h = 0$ or 1 , M_h is just the unit 7-sphere and the sphere bundle is just the Hopf fibration (corresponding to the left or right multiplications of the quaternions, respectively). On the other hand, M_2 is the exotic sphere generating the group $\Theta(7)$ (the set of the orientation preserving diffeomorphism classes of 7-dimensional oriented homotopy spheres), which is isomorphic to the cyclic group \mathbf{Z}_{28} .

It is shown by Eells-Kuiper [EK2] that the homotopy sphere M_h is diffeomorphic to S^7 if and only if the following congruence holds for h ,

$$\frac{h(h-1)}{56} \equiv 0 \pmod{\mathbf{Z}}. \quad (1.1)$$

From now on, we assume that h satisfies (1.1). Then there is a diffeomorphism $\sigma : M_h \rightarrow S^7$. Let $X_{h,\sigma}$ denote the 8 dimensional closed smooth manifold constructed from N_h by attaching the unit disk D^8 by the diffeomorphism $\sigma : \partial(N_h) = M_h \rightarrow \partial(D^8) = S^7$. This is what we call an Eells-Kuiper quaternionic projective plane, first constructed in [EK1].² We remark that when $h = 0$ or 1 , and $\sigma = \text{id}$, $X_{h,\sigma}$ is just the standard quaternionic projective plane \mathbf{HP}^2 . We also mention a deep result due to Kramer and Stolz [KraS] which asserts that the diffeomorphism type of the resulting manifold $X_{h,\sigma}$ does not depend on the choice of the diffeomorphism $\sigma : M_h \rightarrow S^7$.

Let τ_h be the canonical involution on M_h obtained by the fiberwise antipodal involution on S^3 . By [BB, Theorem 1] and the above result of Kramer-Stolz, to prove that $X_{h,\sigma}$ carries an SC^p Riemannian structure, one only need to show that there is a diffeomorphism $\sigma' : M_h \rightarrow S^7$ such that $\tau\sigma' = \sigma'\tau_h$, where τ is the standard antipodal involution of S^7 . Equivalently, one need only to show that the quotient manifold M_h/τ_h is diffeomorphic to \mathbf{RP}^7 . This is the content of the following main result of this paper.

Theorem 1.1. *The involution τ_h on $M_h \cong S^7$ is equivalent to the standard antipodal involution on S^7 . In other words, M_h/τ_h is diffeomorphic to \mathbf{RP}^7 .*

Corollary 1.1. *Every Eells-Kuiper quaternionic projective plane admits an SC^p Riemannian structure.*

²Indeed, Eells and Kuiper showed in [EK1] that the $X_{h,\sigma}$'s are the only 8 dimensional closed smooth manifolds admitting a Morse function with 3 critical points.

The rest of this article is organized as follows. In Section 2, we reduce the proof of Theorem 1.1 to a problem of computing the Eells-Kuiper μ invariant introduced in [EK2]. In Section 3, we carry out the required computation of the μ invariant.

2. THEOREM 1.1 AND THE EELLS-KUIPER μ INVARIANT

As was indicated in [BB, pp. 240], by results of Mayer [Ma], there could only be two possibilities for M_h/τ_h . That is, it is diffeomorphic either to $\mathbf{R}P^7$ or to the connected sum $\mathbf{R}P^7 \# 14M_2$, where $14M_2$ is the connected sum $M_2 \# \cdots \# M_2$ of 14 copies of M_2 .

On the other hand, Milnor [Mi2] showed that the Eells-Kuiper μ invariant of $\mathbf{R}P^7$ and $\mathbf{R}P^7 \# 14M_2$ takes different values. Thus, in order to prove Theorem 1.1, one need only to show that the μ invariant of M_h/τ_h is different from that of $\mathbf{R}P^7 \# 14M_2$.

For completeness, we recall the definition of the Eells-Kuiper μ invariant in our situation. Let M be a 7 dimensional closed oriented spin manifold such that the 4-th cohomology group $H^4(M; \mathbf{R})$ vanishes.³ If M bounds a compact oriented spin manifold N , then the first Pontrjagin class $p_1(N) \in H^4(N, M; \mathbf{Q})$ is well-defined.

Following [EK2, (11)], we define $\mu(M) \in \mathbf{R}/\mathbf{Z}$ by

$$\mu(M) \equiv \frac{p_1^2(N)}{2^7 \times 7} - \frac{\text{Sign}(N)}{2^5 \times 7} \pmod{\mathbf{Z}}, \quad (2.1)$$

where $p_1^2(N)$ denotes the corresponding Pontrjagin number and $\text{Sign}(N)$ is the Signature of N .

Now set $M = M_h$, $N = N_h$. Let $x \in H^4(S^4; \mathbf{Z})$ be the generator. By [Mi1], one has

$$e(\xi_{h,1-h}) = x, \quad p_1(\xi_{h,1-h}) = \pm 2(2h-1)x, \quad (2.2)$$

where $e(\xi_{h,1-h})$ and $p_1(\xi_{h,1-h})$ are the Euler class and the first Pontrjagin class of the sphere bundle $\xi_{h,1-h}$ respectively. Also by [Mi1], one has

$$\text{Sign}(N_h) = 1. \quad (2.3)$$

From (2.2) and (2.3), one deduces as in [Mi1] and [EK2] that

$$\frac{p_1^2(N_h)}{2^7 \times 7} - \frac{\text{Sign}(N_h)}{2^5 \times 7} = \frac{h(h-1)}{56}, \quad (2.4)$$

which is an integer in view of the assumption (1.1).

Recall that by [Mi2], one has $\mu(\mathbf{R}P^7) = \pm \frac{1}{32}$ while $\mu(\mathbf{R}P^7 \# 14M_2) = \pm \frac{1}{32} + \frac{1}{2}$. Thus, in order to prove Theorem 1.1, one need only to prove the following result.

³By the above diffeomorphism type result, it is clear that M_h/τ_h verifies this condition.

Theorem 2.1. *The following identity holds for any integer h verifying (1.1),*

$$\mu(M_h/\tau_h) \equiv \pm \frac{1}{32} \pmod{\mathbf{Z}}. \quad (2.5)$$

Theorem 2.1 will be proved in Section 3

3. A PROOF OF THEOREM 2.1

In this section, we compute $\mu(M_h/\tau_h)$. The obvious difficulty is that one does not find easily an 8 dimensional spin manifold with boundary M_h/τ_h . Instead, we will make use of an intrinsic formula for the μ invariant, which is given by Donnelly [D1] and Kreck-Stolz [KreS].

Indeed, for any 7 dimensional closed oriented spin manifold M with $H^4(M; \mathbf{R}) = 0$, let g^{TM} be a Riemannian metric on TM . Let ∇^{TM} be the associated Levi-Civita connection. Let $p_1(TM, \nabla^{TM})$ be the corresponding first Pontrjagin form (cf. [Z, Section 1.6.2]). Then there is a 3-form $\widehat{p}_1(TM, \nabla^{TM})$ on M such that

$$d\widehat{p}_1(TM, \nabla^{TM}) = p_1(TM, \nabla^{TM}). \quad (3.1)$$

Let D_M (resp. B_M) be the Dirac (resp. Signature) operator associated to g^{TM} . Let $\eta(D_M)$, $\eta(B_M)$ be the Atiyah-Patodi-Singer η invariant of D_M , B_M (cf. [APS]). Let $\overline{\eta}(D_M) = \frac{1}{2}(\dim(\ker D_M) + \eta(D_M))$ be the corresponding reduced η -invariant. Then by [D1] and [KreS], the μ invariant defined in (2.1) can be represented by

$$\mu(M) \equiv \overline{\eta}(D_M) + \frac{\eta(B_M)}{2^5 \times 7} - \frac{1}{2^7 \times 7} \int_M p_1(TM, \nabla^{TM}) \wedge \widehat{p}_1(TM, \nabla^{TM}) \pmod{\mathbf{Z}}. \quad (3.2)$$

Now consider the double covering $M_h \rightarrow M_h/\tau_h$ and lift everything from M_h/τ_h to M_h . We get that

$$\mu(M_h/\tau_h) \equiv \overline{\eta}(P_h D_{M_h}) + \frac{\eta(P_h B_{M_h})}{2^5 \times 7} - \frac{1}{2^8 \times 7} \int_{M_h} p_1(TM_h, \nabla^{TM_h}) \wedge \widehat{p}_1(TM_h, \nabla^{TM_h}) \pmod{\mathbf{Z}}, \quad (3.3)$$

where $P_h = \frac{1}{2}(1 + \tau_h)$ is the canonical projection. Here τ_h denotes the lifted actions on the corresponding vector bundles.

Indeed, recall that M_h is a fiber bundle over S^4 with fiber S^3 . It is the boundary of the unit disk bundle N_h over S^4 , while τ_h is the canonical involution which maps on each fiber by mapping a point to its antipodal. This involution extends canonically to an involution on N_h which we still denote by τ_h . Clearly, the fixed point set of τ_h on N_h is S^4 , the image of the zero section of the disk bundle.

Let g^{TN_h} be a τ_h invariant Riemannian metric on TN_h such that it restricts to g^{TM_h} on $\partial N_h = M_h$ and is of product structure near M_h (the existence of such a metric is clear). Let ∇^{TN_h} be the associated Levi-Civita connection.

It is easy to see (cf. [AB, pp. 487]) that τ_h lifts to an action on the spinor bundle $S(TN_h) = S_+(TN_h) \oplus S_-(TN_h)$ associated to (TN_h, g^{TN_h}) , preserving the corresponding \mathbf{Z}_2 -grading. It induces an action on $S(TM_h) = S_+(TN_h)|_{M_h}$. Moreover, the lifted τ_h action commutes with the Dirac operator $D_{N_h} : \Gamma(S(TN_h)) \rightarrow \Gamma(S(TN_h))$, and thus also commutes with the induced Dirac operator $D_{M_h} : \Gamma(S(TM_h)) \rightarrow \Gamma(S(TM_h))$, which in turn determines a Dirac operator on M_h/τ_h on which one can apply (3.2) and (3.3).

Let $D_{N_h,+} : \Gamma(S_+(TN_h)) \rightarrow \Gamma(S_-(TN_h))$ be the natural restriction of D_{N_h} . By the Atiyah-Patodi-Singer index theorem [APS] and its equivariant extension by Donnelly [D2], one finds

$$\bar{\eta}(P_h D_{M_h}) \equiv \frac{1}{2} \int_{N_h} \hat{A}(TN_h, \nabla^{TN_h}) + \frac{1}{2} \int_{S^4} A_1 \pmod{\mathbf{Z}}, \quad (3.4)$$

where the mod \mathbf{Z} term comes from the Atiyah-Patodi-Singer type index $\text{ind}_{\text{APS}}(P_h D_{N_h,+})$, $\hat{A}(TN_h, \nabla^{TN_h})$ is the Hirzebruch \hat{A} -form associated to ∇^{TN_h} (cf. [Z, Section 1.6.3]) and A_1 is the canonical contribution on the fixed point set. Similarly,

$$\eta(P_h B_{M_h}) = \frac{1}{2} \int_{N_h} L(TN_h, \nabla^{TN_h}) + \frac{1}{2} \int_{S^4} A_2 - \frac{1}{2} \text{Sign}(N_h) - \frac{1}{2} \text{Sign}(N_h, \tau_h), \quad (3.5)$$

where $L(TN_h, \nabla^{TN_h})$ is the Hirzebruch L -form associated to ∇^{TN_h} (cf. [Z, Section 1.6.3]), A_2 is the canonical contribution on the fixed point set and $\text{Sign}(N_h, \tau_h)$ is the notation for the equivariant Signature with respect to τ_h .

By a direct computation, one has

$$\begin{aligned} & \frac{1}{2} \int_{N_h} \hat{A}(TN_h, \nabla^{TN_h}) + \frac{1}{2^6 \times 7} \left(\int_{N_h} L(TN_h, \nabla^{TN_h}) - \text{Sign}(N_h) \right) \\ & - \frac{1}{2^8 \times 7} \int_{M_h} p_1(TM_h, \nabla^{TM_h}) \wedge \hat{p}_1(TM_h, \nabla^{TM_h}) = \frac{p_1^2(N_h)}{2^8 \times 7} - \frac{\text{Sign}(N_h)}{2^6 \times 7}. \end{aligned} \quad (3.6)$$

From (2.4) and (3.3)-(3.6), we find that

$$\mu(M_h/\tau_h) \equiv \frac{h(h-1)}{112} + \frac{1}{2} \int_{S^4} A_1 + \frac{1}{2^6 \times 7} \int_{S^4} A_2 - \frac{\text{Sign}(N_h, \tau_h)}{2^6 \times 7} \pmod{\mathbf{Z}}. \quad (3.7)$$

Now let W_h denote the normal bundle in N_h to the submanifold S^4 , the fixed point set of τ_h . It is clear that τ_h acts on W_h by multiplication by -1 .

By [LM, pp. 267], one finds

$$\int_{S^4} A_1 = \pm \frac{1}{32} \int_{S^4} p_1(W_h) = \pm \frac{(2h-1)}{16}, \quad (3.8)$$

where the second equality follows from (2.2). Similarly, by [LM, pp. 265] and (2.2), one has

$$\int_{S^4} A_2 = \int_{S^4} e(W_h) = 1. \quad (3.9)$$

On the other hand, since S^4 is the fixed point set of τ_h , τ_h preserves $x \in H^4(S^4; \mathbf{Z})$. Thus one has

$$\text{Sign}(N_h, \tau_h) = 1. \quad (3.10)$$

From (3.7)-(3.10), one gets

$$\mu(M_h/\tau_h) \equiv \frac{h(h-1)}{112} \pm \frac{2h-1}{32} \pmod{\mathbf{Z}}. \quad (3.11)$$

We now claim that under the condition (1.1), (2.5) follows from (3.11).

Indeed, under the assumption (1.1), one has $h \equiv 0, 1, 8, 49 \pmod{56\mathbf{Z}}$. Thus we only need to do the case by case checking as follows, where by “ \equiv ” we mean that the congruence is mod \mathbf{Z} .

- (i) For $h = 56k$, then $\frac{h(h-1)}{112} \equiv \frac{k}{2}$, while $\frac{2h-1}{32} \equiv -\frac{1}{32} + \frac{k}{2}$;
- (ii) For $h = 56k + 1$, then $\frac{h(h-1)}{112} \equiv \frac{k}{2}$, while $\frac{2h-1}{32} \equiv \frac{1}{32} + \frac{k}{2}$;
- (iii) For $h = 56k + 8$, one has $\frac{h(h-1)}{112} \equiv \frac{1}{2} + \frac{k}{2}$, while $\frac{2h-1}{32} \equiv -\frac{1}{32} + \frac{1}{2} + \frac{k}{2}$;
- (iv) For $h = 56k + 49$, one has $\frac{h(h-1)}{112} \equiv \frac{k}{2}$, while $\frac{2h-1}{32} \equiv \frac{1}{32} + \frac{k}{2}$.

Combining (i)-(iv) with (3.11), we always have (2.5).

The proof of Theorem 2.1, as well as of Theorem 1.1 and Corollary 1.1 is complete.

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